References

[1]Nayfeh, A. H., and Mook, D. T., *Nonlinear Oscillations,* Wiley-Interscience, New York, 1979, Chap. 7.

[2]Beatty, M. F., and Chow, A. C., "Free Vibrations of a Loaded Rubber String," *International Journal of Non-Linear Mechanics,* Vol. 19, No. 1, 1984, pp. 69–82.

[3]Pastorel, H., and Beaulieu, G., "Nonlinear Vibrations of Guy Cable Systems," *Computers and Structures,* Vol. 21, No. 1/2, 1985, pp. 33–50.

[4]Ali, S. A., "Dynamic Responses of Sagged Cables," *Computers and Structures,* Vol. 23, No. 1, 1986, pp. 51–57.

[5]Tadjbakhsh, I. G., and Wang, Y. M., "Wind-Driven Nonlinear Oscillations of Cables," *Nonlinear Dynamics,* Vol. 1, No. 4, 1990, pp. 265–291.

[6]Fujino, Y., Warnitchai, P., and Pacheco, B. M., "An Experimental and Analytical Study of Autoparametric Resonance in a 3dof Model of Cable-Stayed-Beam," *Nonlinear Dynamics* (to be published).

[7]Papazoglou, V. J., and Mavrakos, S. A., "Nonlinear Cable Response and Model Testing in Water," *Journal of Sound and Vibration,* Vol. 140, No. 1, 1990, pp. 103–115.

[8]Mitsugi, J., and Yasaka, T., "Nonlinear Static and Dynamic Analysis Method of Cable Structures," *AIAA Journal,* Vol. 29, No. 1, 1991, pp. 150–152.

[9]Rao, G. V., and Iyengar, R. N, "Internal Resonance and Nonlinear Response of a Cable under Periodic Excitation," *Journal of Sound and Vibration,* Vol. 149, No. 1, 1991, pp. 25–41.

[10]O'Reilly, O., and Holmes, P., "Nonlinear, Nonplanar and Nonperiodic Vibrations of a String," *Journal of Sound and Vibration* (to be published).

[11]Lee, C. L., and Perkins, N. C., "Nonlinear Oscillations of Suspended Cables Containing a Two-to-One Internal Resonance," *Nonlinear Dynamics* (to be published).

[12]King, A. L., "Oscillations of a Loaded Rubber Band," *American Journal of Physics,* Vol. 42, No. 8, 1974, pp. 699–701.

Beam Deflection

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I. Introduction

THERE have been numerous attempts to improve the classical Euler-Bernoulli beam theory, not to mention the work by Timoshenko.[1] Following Timoshenko, a number of high-order engineering beam theories have been proposed, including works by Duva and Simmonds,[2] Fan and Widera,[3] Rehfield and Murthy,[4] and Rychter.[5] From the viewpoint of the theory of elasticity, Gregory and Wan[6] derive the exact condition that the outer beam solutions must satisfy for decaying states, and Wan[7] offers a correct beam theory. On the other hand, Gregory and Gladwell,[8] Adams and Bogy,[9] and Robert and Keer[10] use the elasticity solution to study stresses, including the edge effects in clamped beams. However, either higher order theories or analyses based on the elasticity theory do not appear simple enough to be used in daily applications. The purpose of this Note is to indicate that a simple modification in the beam theory predicts beam deflections much better than does the Euler-Bernoulli beam theory. The deflections obtained by this modification agree very well with those by the *plate* finite elements. This modification is particularly useful in beams with some kinematic boundary conditions, such as clamped or simply supported. We remark that even in very narrow beams the deflections by the classical Euler-Bernoulli beam theory do not agree very well with those by the plate finite element analysis.

For the modification, we start from the classical plate theory and try to impose geometric constraints from the beginning of derivation. Although beams with kinematic end constraints such as simple or clamped supports are widely used in engineering applications, it appears that in the literature there is no attempt to take explicitly into account such constraints. Since the Euler-Bernoulli theory is derived for beams under pure bending (which are not constrained) and directly extended for beams under other loading and boundary conditions, it can be expected that the theory may not distinguish the different behavior of beams with and without kinematic constraints. Although the result obtained by the present modification is exactly the same as that for wide beams (e.g., see Budynass[11]), the derivation procedures are fundamentally different. It is shown that the present modification can be used for a wide range of beam widths, from very narrow to relatively wide.

II. Analysis for the Proposed Modification

For later use, we just list the Euler-Bernoulli theory for a beam with rectangular cross section (e.g., see Crandall et al.[12]):

$$\frac{\mathrm{d}^2 M_x}{\mathrm{d}x^2} = -p(x) \tag{1}$$

$$M_x = EI\kappa_x \tag{2}$$

$$\kappa_x = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \tag{3}$$

$$EI = \frac{Et^3}{12} \tag{4}$$

The transverse load and deflection are denoted by $p(x)$ and $w(x)$, respectively, and x is the axial coordinate located in the middle plane of a beam. The moment M_x is measured per unit width, and $p(x)$ has the dimension of pressure. E and t represent Young's modulus and the beam thickness. Note that the beam theory is based on the plane-stress assumption in the direction perpendicular to the plane.

Unlike in the derivation of the classical beam theory where pure bending is assumed, we rather start from the Kirchhoff-Love plate theory and consider the effect of kinematic constraints from the beginning of the derivation. The well-known Kirchhoff-Love plate theory is briefly reviewed here (e.g., see Ugural[13]):

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p(x,y) \tag{5}$$

$$M_x = D(\kappa_x + \nu\kappa_y) \tag{6a}$$

$$M_y = D(\kappa_y + \nu\kappa_x) \tag{6b}$$

$$M_{xy} = D(1-\nu)\kappa_{xy} \tag{6c}$$

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2} \tag{7a}$$

$$\kappa_y = -\frac{\partial^2 w}{\partial y^2} \tag{7b}$$

$$\kappa_{xy} = -\frac{\partial^2 w}{\partial x \partial y} \tag{7c}$$

$$D = \frac{Et^3}{12(1-\nu^2)} \tag{8}$$

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In Eqs. (5–7), M and κ stand for moment and curvature, respectively, and ν is Poisson's ratio. The coordinate y represents the width direction. The sign convention is the same as that used in Ugural.[13] The plate dimensions in x and y will be denoted by L and b. (Thus, the width of a beam would be given by b.)

It is easily seen that a beam can be viewed as a special plate element where the width b approaches 0. Since the variation of the applied load in y can be neglected, the load can be assumed to be a function of x only. Thereby, we can assume

$$\kappa_{xy}(x,y) = -\frac{\partial^2 w(x,y)}{\partial x \partial y} \equiv 0 \tag{9}$$

When a beam is simply supported or clamped somewhere in the beam (say, $x = x_1$), the deflection at that location is identically zero over the beam width:

$$w(x,y) = 0 \qquad \text{at } x = x_1 \tag{10}$$

From Eq. (10), it is straightforward to obtain

$$\kappa_y(x,y) = -\frac{\partial^2 w(x,y)}{\partial y^2} = 0 \qquad \text{at } x = x_1 \tag{11}$$

Now, we differentiate Eq. (9) once with respect to y and integrate it over x. Using the condition of Eq. (11), we find that everywhere

$$\kappa_y(x,y) = -\frac{\partial^2 w(x,y)}{\partial y^2} \equiv 0 \tag{12}$$

Substituting Eqs. (9) and (12) into Eqs. (5) and (6) gives

$$\frac{\partial^4 w}{\partial x^4} + \nu \frac{\partial^4 w}{\partial x^2 \partial y^2} = \frac{p(x,y)}{D} \tag{13}$$

The substitution of Eq. (12) into Eq. (13) results in

$$\frac{\partial^4 w}{\partial x^4} = \frac{p(x,y)}{D} \tag{14}$$

If we summarize the result (with the y dependence off),

$$\frac{d^2 M_x(x)}{dx^2} = -p(x) \tag{15}$$

$$M_x = D\kappa_x \tag{16}$$

$$\kappa_x = -\frac{d^2 w}{dx^2} \tag{17}$$

$$D = \frac{Et^3}{12(1-\nu^2)} \tag{18}$$

$$M_y = \nu M_x, \qquad M_{xy} = 0, \qquad \kappa_y = \kappa_{xy} = 0 \tag{19}$$

Since the derivation of the previous equations utilizes $\kappa_y = 0$ throughout the beam, this assumption may be called the *plane-curvature* assumption. (More precisely, the condition may be named the *thickness-plane-stress* and *width-plane-curvature* condition.)

The limitation of the present result is that, when there is no kinematic constraint as in pure bending, the result is not as accurate as in beams with kinematic constraints. In this instance, the usual assumption for very thin beams is more appropriate:

$$M_y(x,y) = 0 \tag{20}$$

Substituting Eqs. (9) and (20) into Eq. (5) gives Eqs. (1–4), which is the Euler-Bernoulli theory. Because of Eq. (20), the Euler-Bernoulli theory may be viewed as a theory based on the *plane-moment* assumption in contrast with the plane-curvature assumption.

Budynas[11] gives exactly the same equations as Eqs. (15–19) for *wide* beams, which are based on the plane-strain assumption. However, the physics behind the two results is different. When the plane-strain assumption is applied in the width direction, the corresponding theory is appropriate for relatively wide beams regardless of geometric constraints of beams. On the other hand, the present modification based on the plane-curvature assumption through the width direction gives good results for beams with some kinematic constraints regardless of the beam width. (Obviously, we do not consider very wide beams, which behave as plates.) The present modification is useful in a wide range of beam widths from very narrow to wide, as long as there is any kinematic constraint.

Since beams with certain kinds of supports are common, the present modification appears to be practically useful. In the next section, we check the validity of the present modification in terms of numerical examples.

III. Numerical Examples

As the first example, a simply supported beam of length L, thickness t, and width b under a uniform transverse load distribution p_0 is considered. For $L/t = 20$ and $b/t = 1$, the transverse deflection $w(x)$ is plotted in Fig. 1 as a function of x for various Poisson's ratios ν. The results are given only for $0 \le x/t \le 10$ due to symmetry. The numerical results from the finite element calculation by ADINA[14] are marked by circles in Fig. 1. In ADINA, the plate finite elements are employed to analyze all of the problems considered in this work. The convergence of the numerical results has been checked.

As shown in Fig. 1, the present results, based on the plane-curvature assumption, agree excellently with the finite element results. Since the ADINA version used was not capable of handling the incompressible limit ($\nu = 0.5$), $\nu = 0.499$ was used to simulate the limiting case. We note that the Euler-Bernoulli theory gives the same results regardless of Poisson's ratios,

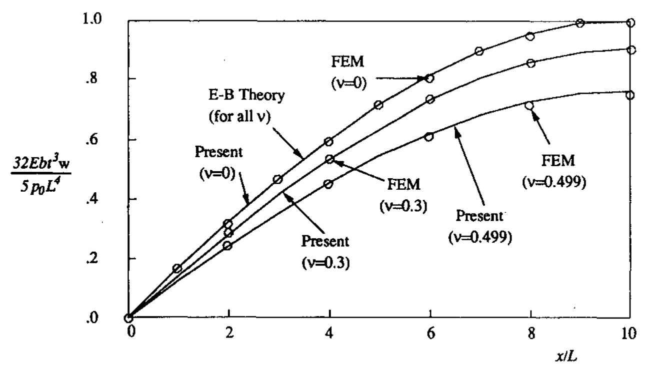

Fig. 1 Transverse deflection of a simply supported beam under a uniform load for different Poisson's ratios ν ($L/t = 20$, $b/t = 1$).

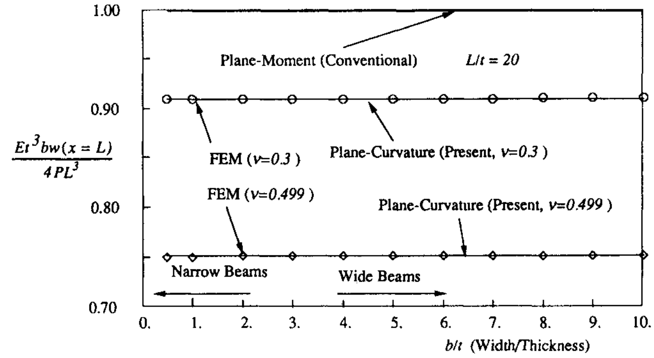

Fig. 2 Deflection at the tip ($x = L$) of the cantilever beam of length L ($L/t = 20$).

whereas the present modification and the plate finite element analysis can capture the variation of beam deflections depending on Poisson's ratios.

The second example is a cantilever beam under a point load P (more precisely, the line load through the beam thickness) at the end $x = L$ of the beam. For varying aspect ratios of the width b to the thickness $t(L/t = 20)$, the normalized deflection w at the tip $x = L$ of the neutral axis is given in Fig. 2. Different Poisson's ratios are also used. Note again that the Euler-Bernoulli theory produces the same result (thick line) for all ranges of Poisson's ratios.

It is apparent that the present results based on the plane-curvature assumption are excellent in a wide range of b/t for different Poisson's ratios. For instance, when $\nu = 0.499$ and $b/t = 0.5$, the tip displacement by the usual beam theory is about 33.3% off from the ADINA result, whereas the tip displacement by the present beam theory is only 0.08% off from it. It is remarked that the beam under consideration is very narrow.

IV. Conclusions

It is found that the simple modification in the beam theory can predict beam deflections considerably better than can the Euler-Bernoulli beam theory for a wide range of the beam widths. Possibly kinematic constraints of beams are explicitly considered in carrying out the modification. Because beams with some kinematic constraints such as simple or clamped supports are common, the present modification appears practically useful in predicting the behavior of beams more realistically.

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References

[1]Timoshenko, S. P., "On the Correction for Shear of the Differential Equation for Transverse Vibrations of Prismatic Bars," *Philosophical Magazine*, Vol. 41, Ser. 6, 1921, pp. 744–746.

[2]Duva, J. M., and Simmonds, J. G., "Elementary, Static Beam Theory Is as Accurate as You Please," *Journal of Applied Mechanics*, Vol. 57, March 1990, pp. 134–137.

[3]Fan, H., and Widera, G. E. O., "Refined Beam Theory Based on the Asymptotic Expansion Approach," *AIAA Journal*, Vol. 29, No. 3, 1991, pp. 444–449.

[4]Rehfield, L., and Murthy, P. L. N., "Toward a New Engineering Theory of Bending: Fundamentals," *AIAA Journal*, Vol. 20, No. 4, 1982, pp. 693–699.

[5]Rychter, Z., "A Simple and Accurate Beam Theory," *Acta Mechanica*, Vol. 75, 1988, pp. 57–62.

[6]Gregory, R. D., and Wan, F. Y. M., "Decaying States of Plane Strain in a Semi-Infinite Strip and Boundary Conditions for Plate Theory," *Journal of Elasticity*, Vol. 14, 1984, pp. 27–64.

[7]Wan, F. Y. M., "Interior Solutions for Slender and Thin Elastic Bodies with Prescribed Displacement End Data," *Analytical and Computational Models of Shells,* edited by A. K. Noor, T. Belytshko, and J. C. Simo, American Society of Mechanical Engineers, New York, 1989, pp. 55–70.

[8]Gregory, R. D., and Gladwell, I., "The Cantilever Beam under Tension, Bending or Flexure at Infinity," *Journal of Elasticity,* Vol. 12, 1982, pp. 317–343.

[9]Adams, G. G., and Bogy, D. B., "The Plane Solution for Bending of Joined Dissimilar Elastic Semi-Infinite Strips," *International Journal of Solids and Structures*, Vol. 12, No. 3, 1976, pp. 239–249.

[10]Robert, M., and Keer, L. M., "An Elastic Circular Cylinder with Prescribed Displacements at the Ends-Asymmetric Case," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 40, Pt. 3, 1987, pp. 365–381.

[11]Budynas, R. G., *Advanced Strength and Applied Stress Analysis*, McGraw-Hill, New York, 1977, Chap. 3.

[12]Crandall, S. H., Dahl, N. C., and Lardner, T. J., *An Introduction to the Mechanics of Solids*, 2nd ed., McGraw-Hill, New York, 1972, Chap. 7.

[13]Ugural, A. C., *Stresses in Plates and Shells*, McGraw-Hill, New York, 1981, Chap. 1.

[14]Anon., *ADINA System Verification Manual*, ADINA Engineering Inc., Watertown, MA, 1983.